

# PARABOLIC PROBLEMS AND INTERPOLATION WITH A FUNCTION PARAMETER

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**ABSTRACT.** We give an application of interpolation with a function parameter to parabolic differential operators. We introduce the refined anisotropic Sobolev scale that consists of some Hilbert function spaces of generalized smoothness. The latter is characterized by a real number and a function varying slowly at infinity in Karamata's sense. This scale is connected with anisotropic Sobolev spaces by means of interpolation with a function parameter. We investigate a general initial-boundary value parabolic problem in the refined Sobolev scale. We prove that the operator corresponding to this problem sets isomorphisms between appropriate spaces pertaining to this scale.

## 1. INTRODUCTION

In the theory of partial differential equations, the question about regularity properties of solutions to equations is of great importance. As a rule, an answer to this question is given in the form of sufficient conditions for the solutions to belong to certain function spaces. The latter depend on a finite collection of number parameters and form a scale of spaces. The more finely the scale is calibrated by these parameters, the more precise and complete an information about the solution properties will be.

Basically, researchers use the two scales formed by Sobolev spaces and Hölder-Zygmund spaces respectively. For these scales, we have the theory of general elliptic boundary-value problems [1, 2, 12, 16, 32, 35, 38] and parabolic initial-boundary value problems [8, 10, 11, 15, 17] (also see the surveys [3, 9] and the bibliography given therein). However, these scales proved to be coarse for some applications to differential operators [12, 13, 24, 25, 26].

In this connection, of interest are spaces for which a function parameter, not a number, serves as an smoothness index. They are called spaces of generalized smoothness. Important classes of such spaces were introduced and investigated by L. Hörmander [12] and L. R. Volevich, B. P. Paneah [37]. L. Hörmander [12, 13] gave

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a systematical application of these spaces to the research on regularity properties of solutions to hypoelliptic equations. Nowadays spaces of generalized smoothness are used in various investigations [14, 30, 31, 36].

As regards applications – specifically, to the spectral theory of differential equations – scales of Hilbert function spaces are especially important. Until recently only Hilbert Sobolev scale and its various weighted or anisotropic modifications have been used in the theory of differential equations. Lately V. A. Mikhailets and the second author [18–24, 27–29] elaborated the theory of general elliptic differential operators and elliptic boundary-value problems in the Hilbert scales formed by the Hörmander spaces

$$(1.1) \quad H^{s,\varphi} := B_{2,\mu} \quad \text{for} \quad \mu(\xi) := (1 + |\xi|^2)^{s/2} \varphi((1 + |\xi|^2)^{1/2}).$$

Here the number parameter  $s$  is real, whereas the function parameter  $\varphi$  varies slowly at infinity in J. Karamata's sense. For example,  $\varphi$  can be logarithmic function, its iterations, each of their powers, and multiplications of these functions. The class of the spaces (1.1) contains the Sobolev scale  $\{H^s\} = \{H^{s,1}\}$  and is attached to it by means of  $s$  but is calibrated more finely than the Sobolev scale. The number parameter  $s$  sets the main (power) smoothness, while the function parameter  $\varphi$  defines a supplementary (subpower) smoothness. The latter may give the broader or narrower space  $H^{s,\varphi}$  as compared with  $H^s$ .

The spaces (1.1) form the refined Sobolev scale. It possesses an important interpolation property. Namely, each space (1.1) can be obtained by interpolation, with an appropriate function parameter, of a certain couple of Sobolev spaces (see [22, Sec. 3.2] or [24, Sec. 1.3.4]). This parameter is a function that varies slowly of an index  $\theta \in (0, 1)$  at infinity, in the sense of J. Karamata. The refined Sobolev scale is closed with respect to the interpolation with these function parameters.

For linear operators, their boundedness and Fredholm property will be preserved when the interpolation of the corresponding spaces is done. This fact allowed the authors by [24] to transfer, to the full extent, the classical (Sobolev) theory of elliptic partial differential equations to the case of the refined Sobolev scale. As an application of this theory, we mention the theorems on convergence almost everywhere and uniformly of spectral expansions in eigenfunctions of self-adjoint positive elliptic differential operators (see [24, Sec. 2.3] or [27, Sec. 7.2]). It is essential for this theorems that the smoothness index is a function parameter (also see [25, 26]).

Note that, the interpolation with a power parameter  $\varphi(t) \equiv t^\theta$  plays an important role in the Sobolev theory of partial differential equations, the exponent  $\theta$  serving as a number parameter of the interpolation. A systematic application of this interpolation to various classes of differential operators is given by J.-L. Lions, E. Magenes [16, 17], and H. Triebel [34, 35].

In this paper we give an application of interpolation with a function parameter to parabolic partial differential equations. They differ from elliptic equations in disparity of independent variables (temporal and spatial), which implies the need to use anisotropic function spaces. Therefore we introduce a certain anisotropic

analog of the refined Sobolev scale. For this analog, we establish a theorem on the isomorphisms that are realized by the operator corresponding to an initial-boundary value problem for a parabolic equation of an arbitrary even order. This theorem will be proved by means of the interpolation with a function parameter between anisotropic Sobolev spaces. We use regularly varying functions as interpolation parameters. In order that our reasoning should be more transparent, we restrict ourselves to the two-dimensional case and assume that the initial conditions are homogeneous.

The paper consists of six sections. Section 1 is Introduction. In Section 2, we state an initial-boundary value problem for a general parabolic equation given in a rectangular planar domain. Here we also formulate the main result of the paper, the theorem on isomorphisms. In Section 3, we introduce and discuss the refined anisotropic Sobolev scale over  $\mathbb{R}^2$  and its analogs for the rectangular domain. These analogs conform to the parabolic problem under consideration. In Section 4, we give necessary facts about the interpolation with a function parameter between general Hilbert spaces. Main Theorem is proved in Section 5. Here we previously deduce necessary interpolation formulas, which connect the introduced scale with anisotropic Sobolev spaces. In the last section, 6, we indicate some applications and generalizations of Main Theorem.

## 2. STATEMENT OF THE PROBLEM AND MAIN RESULT

Let  $\Omega := (0, l) \times (0, \tau)$ , where positive numbers  $l$  and  $\tau$  are chosen arbitrarily. Consider the following linear parabolic initial-boundary value problem in the open rectangle  $\Omega$ :

$$(2.1) \quad \begin{aligned} & A(x, t, D_x, \partial_t)u(x, t) \\ & \equiv \sum_{\alpha+2b\beta \leq 2m} a^{\alpha, \beta}(x, t) D_x^\alpha \partial_t^\beta u(x, t) = f(x, t) \quad \text{in } \Omega, \end{aligned}$$

$$(2.2) \quad \begin{aligned} & B_{j,0}(t, D_x, \partial_t)u(x, t)|_{x=0} \\ & \equiv \sum_{\alpha+2b\beta \leq m_j} b_{j,0}^{\alpha, \beta}(t) D_x^\alpha \partial_t^\beta u(x, t)|_{x=0} = g_{j,0}(t) \quad \text{and} \end{aligned}$$

$$(2.3) \quad \begin{aligned} & B_{j,1}(t, D_x, \partial_t)u(x, t)|_{x=l} \\ & \equiv \sum_{\alpha+2b\beta \leq m_j} b_{j,1}^{\alpha, \beta}(t) D_x^\alpha \partial_t^\beta u(x, t)|_{x=l} = g_{j,1}(t) \\ & \quad \text{for } 0 < t < \tau \quad \text{and } j = 1, \dots, m, \end{aligned}$$

$$(2.4) \quad \left. \frac{\partial^k u(x, t)}{\partial t^k} \right|_{t=0} = 0 \quad \text{for } 0 < x < l \quad \text{and } k = 0, \dots, \varkappa - 1.$$

Here  $b$ ,  $m$ , and all  $m_j$  are arbitrarily fixed integers such that  $m \geq b \geq 1$ ,  $\varkappa := m/b \in \mathbb{Z}$ , and  $m_j \geq 0$ . All coefficients of the partial differential expressions  $A := A(x, t, D_x, \partial_t)$  and  $B_{j,k} := B_{j,k}(t, D_x, \partial_t)$ , with  $j \in \{1, \dots, m\}$  and  $k \in \{0, 1\}$ , are

supposed to be complex-valued and infinitely smooth functions; namely,  $a^{\alpha,\beta} \in C^\infty(\overline{\Omega})$  and  $b_{j,k}^{\alpha,\beta} \in C^\infty[0, \tau]$ , where  $\overline{\Omega} := [0, l] \times [0, \tau]$  as usual. We use the notation  $D_x := i\partial/\partial x$  and  $\partial_t := \partial/\partial t$  for partial derivatives and take summation over the integer-valued indexes  $\alpha, \beta \geq 0$  satisfying the conditions indicated.

Recall [4, § 9, Subsec. 1] that the initial-boundary value problem (2.1)–(2.4) is said to be parabolic in  $\Omega$  if the following three conditions are fulfilled:

(i) Given any  $x \in [0, l]$ ,  $t \in [0, \tau]$ ,  $\xi \in \mathbb{R}$ , and  $p \in \mathbb{C}$  with  $\operatorname{Re} p \geq 0$ , we have

$$\begin{aligned} & A^{(0)}(x, t, \xi, p) \\ & \equiv \sum_{\alpha+2\beta=2m} a^{\alpha,\beta}(x, t) \xi^\alpha p^\beta \neq 0 \quad \text{whenever} \quad |\xi| + |p| \neq 0. \end{aligned}$$

(ii) Let  $x \in \{0, l\}$ ,  $t \in [0, \tau]$ , and  $p \in \mathbb{C} \setminus \{0\}$  with  $\operatorname{Re} p \geq 0$  be arbitrary. Then the polynomial  $A^{(0)}(x, t, \xi, p)$  in  $\xi \in \mathbb{C}$  has  $m$  roots  $\xi_j^+(x, t, p)$ ,  $j = 1, \dots, m$ , with positive imaginary part and  $m$  roots with negative imaginary part provided that each root is taken the number of times equal to its multiplicity.

(iii) Assume that  $x$ ,  $t$ , and  $p$  are the same as ones considered in (ii). Let  $k := 0$  if  $x = 0$ , and let  $k := 1$  if  $x = l$ . Then the polynomials

$$B_{j,k}^{(0)}(t, \xi, p) \equiv \sum_{\alpha+2\beta=m_j} b_{j,k}^{\alpha,\beta}(t) \xi^\alpha p^\beta, \quad j = 1, \dots, m,$$

in  $\xi$  are linearly independent modulo

$$\prod_{j=1}^m (\xi - \xi_j^+(x, t, p)).$$

Consider the linear mapping

$$\begin{aligned} (2.5) \quad C_+^\infty(\overline{\Omega}) & \ni u \mapsto (Au, Bu) \\ & := (Au, B_{1,0}u, B_{1,1}u, \dots, B_{m,0}u, B_{m,1}u) \in C_+^\infty(\overline{\Omega}) \times (C_+^\infty[0, \tau])^{2m}, \end{aligned}$$

which is associated with the parabolic problem (2.1)–(2.4). Here

$$\begin{aligned} C_+^\infty(\overline{\Omega}) & := \{w \upharpoonright \overline{\Omega} : w \in C^\infty(\mathbb{R}^2), \operatorname{supp} w \subseteq \mathbb{R} \times [0, \infty)\} \\ & = \{u \in C^\infty(\overline{\Omega}) : \partial_t^\beta u(x, t)|_{t=0} = 0 \text{ for all } \beta \in \mathbb{N} \cup \{0\}, x \in [0, l]\} \end{aligned}$$

and

$$\begin{aligned} C_+^\infty[0, \tau] & := \{h \upharpoonright [0, \tau] : h \in C^\infty(\mathbb{R}), \operatorname{supp} h \subseteq [0, \infty)\} \\ & = \{v \in C^\infty[0, \tau] : v^{(\beta)}(0) = 0 \text{ for all } \beta \in \mathbb{N} \cup \{0\}\}. \end{aligned}$$

In the paper, functions (and distributions) are supposed to be complex-valued unless otherwise stated.

The mapping (2.5) sets a one-to-one correspondence between the spaces  $C_+^\infty(\overline{\Omega})$  and  $C_+^\infty(\overline{\Omega}) \times (C_+^\infty[0, \tau])^{2m}$  (see Remark 3.3 below). Our purpose is to show that this mapping extends by continuity to an isomorphism between appropriate couples

of Hilbert function spaces of generalized smoothness. Namely, we will prove the following result.

Let  $\sigma_0$  be the smallest integer such that

$$\sigma_0 \geq 2m, \quad \sigma_0 \geq m_j + 1 \text{ for all } j \in \{1, \dots, m\}, \quad \text{and} \quad \frac{\sigma_0}{2b} \in \mathbb{Z}.$$

Note, if  $m_j \leq 2m - 1$  for every  $j \in \{1, \dots, m\}$ , then  $\sigma_0 = 2m$ .

**Main Theorem.** *Let a real number  $\sigma > \sigma_0$  and function parameter  $\varphi \in \mathcal{M}$  be chosen arbitrarily. Then the mapping (2.5) extends uniquely (by continuity) to an isomorphism*

$$(2.6) \quad \begin{aligned} (A, B) : H_+^{\sigma, \sigma/(2b), \varphi}(\Omega) \\ \leftrightarrow H_+^{\sigma-2m, (\sigma-2m)/(2b), \varphi}(\Omega) \oplus \bigoplus_{j=1}^m (H_+^{(\sigma-m_j-1/2)/(2b), \varphi}(0, \tau))^2. \end{aligned}$$

The class  $\mathcal{M}$  and the Hilbert function spaces occurring in (2.6) will be defined in the next section. These spaces form the refined Sobolev scales.

If  $\varphi \equiv 1$ , then the operator (2.6) acts between Sobolev spaces. In this case, this theorem was proved by M. S. Agranovich and M. I. Vishik [4, Theorem 11.1] on the assumption that  $\sigma/(2b) \in \mathbb{Z}$ . Their result includes the limiting case of  $\sigma = \sigma_0$  and relates to general parabolic problems with nonhomogeneous initial conditions.

### 3. REFINED SOBOLEV SCALES

In this section we will introduce and discuss the function spaces used in the statement of Main Theorem. The regularity properties of the distributions belonging to these spaces are characterized by two number parameters and a function parameter. The latter runs over a certain function class  $\mathcal{M}$ , which is defined as follows.

The class  $\mathcal{M}$  consists of all functions  $\varphi : [1, \infty) \rightarrow (0, \infty)$  such that

- a)  $\varphi$  is Borel measurable on  $[1, \infty)$ ;
- b) both the functions  $\varphi$  and  $1/\varphi$  are bounded on each compact interval  $[1, b]$ , with  $1 < b < \infty$ ;
- c)  $\varphi$  is a slowly varying function at infinity in the sense of J. Karamata; i.e.,

$$(3.1) \quad \lim_{r \rightarrow \infty} \frac{\varphi(\lambda r)}{\varphi(r)} = 1 \quad \text{for every } \lambda > 0.$$

*Remark 3.1.* The theory of slowly varying functions is set forth in the monographs [7, 33]. We give an important and standard example of functions satisfying (3.1) if we put

$$(3.2) \quad \varphi(r) := (\log r)^{\theta_1} (\log \log r)^{\theta_2} \dots (\underbrace{\log \dots \log r}_{k \text{ times}})^{\theta_k} \quad \text{for } r \gg 1,$$

where the parameters  $k \in \mathbb{N}$  and  $\theta_1, \theta_2, \dots, \theta_k \in \mathbb{R}$  are chosen arbitrarily. The functions (3.2) form the logarithmic multiscale, which has a number of applications

in the theory of function spaces. Some other examples of slowly varying functions can be found in [7, Sec. 1.3.3] and [24, Sec. 1.2.1].

Let  $s \in \mathbb{R}$ ,  $\varphi \in \mathcal{M}$ , and  $\gamma := 1/(2b)$ . By definition, the linear space  $H^{s,s\gamma,\varphi}(\mathbb{R}^2)$  consists of all tempered distributions  $w \in \mathcal{S}'(\mathbb{R}^2)$  such that their Fourier transform  $\tilde{w}$  (in two variables) is locally Lebesgue integrable over  $\mathbb{R}^2$  and satisfies the condition

$$(3.3) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r_{\gamma}^{2s}(\xi, \eta) \varphi^2(r_{\gamma}(\xi, \eta)) |\tilde{w}(\xi, \eta)|^2 d\xi d\eta < \infty.$$

Here and below we use the notation

$$r_{\gamma}(\xi, \eta) := (1 + |\xi|^2 + |\eta|^{2\gamma})^{1/2} \quad \text{for each } \xi, \eta \in \mathbb{R}.$$

The space  $H^{s,s\gamma,\varphi}(\mathbb{R}^2)$  is endowed with the inner product

$$(w_1, w_2)_{H^{s,s\gamma,\varphi}(\mathbb{R}^2)} := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r_{\gamma}^{2s}(\xi, \eta) \varphi^2(r_{\gamma}(\xi, \eta)) \tilde{w}_1(\xi, \eta) \overline{\tilde{w}_2(\xi, \eta)} d\xi d\eta,$$

where  $w_1, w_2 \in H^{s,s\gamma,\varphi}(\mathbb{R}^2)$ . It induces the norm

$$\|w\|_{H^{s,s\gamma,\varphi}(\mathbb{R}^2)} := (w, w)_{H^{s,s\gamma,\varphi}(\mathbb{R}^2)}^{1/2},$$

which is equal to the square root of the left-hand side of inequality (3.3).

Note that  $H^{s,s\gamma,\varphi}(\mathbb{R}^2)$  is the inner product Hörmander space  $\mathcal{B}_{2,\mu}(\mathbb{R}^2)$  which corresponds to the function parameter

$$\mu(\xi, \eta) := r_{\gamma}^s(\xi, \eta) \varphi(r_{\gamma}(\xi, \eta)) \quad \text{for } \xi, \eta \in \mathbb{R}.$$

We refer the reader to the monographs by L. Hörmander [12, Sec. 2.2], [13, Sec. 10.1], and to the paper by L. R. Volevich and B. P. Paneah [37], where such spaces are investigated systematically. It follows from properties of Hörmander spaces that the space  $H^{s,s\gamma,\varphi}(\mathbb{R}^2)$  is Hilbert and separable, is embedded continuously in  $\mathcal{S}'(\mathbb{R}^2)$ , and that the set  $C_0^{\infty}(\mathbb{R}^2)$  is dense in  $H^{s,s\gamma,\varphi}(\mathbb{R}^2)$ .

*Remark 3.2.* We use conventional notation for main function spaces. So,  $\mathcal{S}'(\mathbb{R}^n)$  denotes the linear topological L. Schwartz space of all tempered distributions given in  $\mathbb{R}^n$ , with  $n \in \mathbb{N}$ . If  $G$  is an open subset of  $\mathbb{R}^n$  (in particular,  $G = \mathbb{R}^n$ ), then  $C_0^{\infty}(G)$  stands for the class of all functions  $w \in C^{\infty}(\mathbb{R}^n)$  such that their support is a compact subset of  $G$ . We may naturally identify a function  $w \in C_0^{\infty}(G)$  with its restriction to  $G$ ; from the context it will always be understood on which set —  $\mathbb{R}^n$  or  $G$  — the function  $w$  is considered. The designation  $L_2(G, d\mu)$  refers to the Hilbert space of all functions that are square integrable over  $G$  with respect to a Radon measure  $\mu$ . Specifically, if  $\mu$  is the Lebesgue measure, then we omit  $d\mu$  and write  $L_2(G)$ .

If  $\varphi(r) \equiv 1$ , then  $H^{s,s\gamma,\varphi}(\mathbb{R}^2)$  becomes the anisotropic Sobolev space of order  $(s, s\gamma)$ ; we denote this space by  $H^{s,s\gamma}(\mathbb{R}^2)$ . Note that, in the case where  $s, s\gamma \in \mathbb{N}$

the space  $H^{s,s\gamma}(\mathbb{R}^2)$  consists of all functions  $w(x, t)$  such that  $w$ ,  $D_x^s w$ , and  $\partial_t^{s\gamma} w$  are square integrable over  $\mathbb{R}^2$ , providing the partial derivatives are understood in the sense of the theory of distributions. In this case, we have the equivalence of Hilbert norms

$$(3.4) \quad \|w\|_{H^{s,s\gamma}(\mathbb{R}^2)} \asymp \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|w(x, t)|^2 + |D_x^s w(x, t)|^2 + |\partial_t^{s\gamma} w(x, t)|^2) dx dt \right)^{1/2}.$$

Every space  $H^{s,s\gamma,\varphi}(\mathbb{R}^2)$ , with  $s \in \mathbb{R}$  and  $\varphi \in \mathcal{M}$ , is closely connected to anisotropic Sobolev spaces. Specifically, we have the continuous and dense embeddings

$$(3.5) \quad H^{s_1, s_1\gamma}(\mathbb{R}^2) \hookrightarrow H^{s, s\gamma, \varphi}(\mathbb{R}^2) \hookrightarrow H^{s_0, s_0\gamma}(\mathbb{R}^2) \quad \text{whenever } s_0 < s < s_1.$$

They follow from the next property of  $\varphi \in \mathcal{M}$ : for each  $\varepsilon > 0$  there exists a number  $c = c(\varepsilon) \geq 1$  such that  $c^{-1}r^{-\varepsilon} \leq \varphi(r) \leq cr^\varepsilon$  for all  $r \geq 1$  (see [33, Sec. 1.5, Subsec. 1]).

Consider the class of Hilbert function spaces

$$(3.6) \quad \{H^{s,s\gamma,\varphi}(\mathbb{R}^2) : s \in \mathbb{R}, \varphi \in \mathcal{M}\}.$$

Owing to the embeddings (3.5), we may assert that in (3.6) the function parameter  $\varphi$  defines a supplementary (subpower) smoothness with respect to the basic (power) anisotropic  $(s, s\gamma)$ -smoothness. Specifically, if  $\varphi(r) \rightarrow \infty$  [ $\varphi(r) \rightarrow 0$ ] as  $r \rightarrow \infty$ , then  $\varphi$  defines a positive [negative] supplementary smoothness. In other words,  $\varphi$  refines the power smoothness  $(s, s\gamma)$ .

Therefore we will naturally call (3.6) the refined anisotropic Sobolev scale over  $\mathbb{R}^2$ ; here  $\gamma$  serves as an anisotropy parameter.

Using this scale, let us introduce some function spaces related to the parabolic problem under consideration. As before,  $s \in \mathbb{R}$  and  $\varphi \in \mathcal{M}$ . We put

$$H_+^{s,s\gamma,\varphi}(\mathbb{R}^2) := \{w \in H^{s,s\gamma,\varphi}(\mathbb{R}^2) : \text{supp } w \subseteq \mathbb{R} \times [0, \infty)\}.$$

The linear space  $H_+^{s,s\gamma,\varphi}(\mathbb{R}^2)$  is endowed with the inner product and norm in  $H^{s,s\gamma,\varphi}(\mathbb{R}^2)$ . The space  $H_+^{s,s\gamma,\varphi}(\mathbb{R}^2)$  is complete (Hilbert) because of the continuous embedding

$$H^{s,s\gamma,\varphi}(\mathbb{R}^2) \hookrightarrow \mathcal{S}'(\mathbb{R}^2).$$

Next, we define the normed linear space

$$(3.7) \quad \begin{aligned} H_+^{s,s\gamma,\varphi}(\Omega) &:= \{w \upharpoonright \Omega : w \in H_+^{s,s\gamma,\varphi}(\mathbb{R}^2)\}, \\ \|u\|_{H_+^{s,s\gamma,\varphi}(\Omega)} &:= \inf \{\|w\|_{H^{s,s\gamma,\varphi}(\mathbb{R}^2)} : w \in H_+^{s,s\gamma,\varphi}(\mathbb{R}^2), w = u \text{ in } \Omega\}, \end{aligned}$$

with  $u \in H_+^{s,s\gamma,\varphi}(\Omega)$ . In other words,  $H_+^{s,s\gamma,\varphi}(\Omega)$  is the factor space of the space  $H_+^{s,s\gamma,\varphi}(\mathbb{R}^2)$  by its subspace

$$(3.8) \quad H_Q^{s,s\gamma,\varphi}(\mathbb{R}^2) := \{w \in H^{s,s\gamma,\varphi}(\mathbb{R}^2) : \text{supp } w \subseteq Q := \mathbb{R} \times [0, \infty) \setminus \Omega\}.$$

Hence, the space  $H_+^{s,s\gamma,\varphi}(\Omega)$  is Hilbert. The norm (3.7) is induced by the inner product

$$(u_1, u_2)_{H_+^{s,s\gamma,\varphi}(\Omega)} := (w_1 - \Upsilon w_1, w_2 - \Upsilon w_2)_{H^{s,s\gamma,\varphi}(\mathbb{R}^2)},$$

where  $w_j \in H^{s,s\gamma,\varphi}(\mathbb{R}^2)$ ,  $w_j = u_j$  in  $\Omega$  for each  $j \in \{1, 2\}$ , and  $\Upsilon$  is the orthogonal projector of the space  $H_+^{s,s\gamma,\varphi}(\mathbb{R}^2)$  onto its subspace (3.8).

Note that both Hilbert spaces  $H_+^{s,s\gamma,\varphi}(\mathbb{R}^2)$  and  $H_+^{s,s\gamma,\varphi}(\Omega)$  are separable. The set  $C_0^\infty(\mathbb{R} \times (0, \infty))$  is dense in  $H_+^{s,s\gamma,\varphi}(\mathbb{R}^2)$  [37, Lemma 3.3]; this implies the density of  $C_+^\infty(\overline{\Omega})$  in  $H_+^{s,s\gamma,\varphi}(\Omega)$ .

It remains to introduce the function spaces in which the right-hand sides of the boundary-value conditions (2.2) and (2.3) are considered. Let  $s \in \mathbb{R}$  and  $\varphi \in \mathcal{M}$ . By definition, the linear space  $H^{s,\varphi}(\mathbb{R})$  consists of all tempered distributions  $h \in \mathcal{S}'(\mathbb{R})$  such that their Fourier transform  $\widehat{h}$  is locally Lebesgue integrable over  $\mathbb{R}$  and satisfies the condition

$$\int_{-\infty}^{\infty} \langle \xi \rangle^{2s} \varphi^2(\langle \xi \rangle) |\widehat{h}(\xi)|^2 d\xi < \infty.$$

Here, as usual,  $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$  is the smooth modulus of  $\xi \in \mathbb{R}$ . The space  $H^{s,\varphi}(\mathbb{R})$  is endowed with the inner product

$$(h_1, h_2)_{H^{s,\varphi}(\mathbb{R})} := \int_{-\infty}^{\infty} \langle \xi \rangle^{2s} \varphi^2(\langle \xi \rangle) \widehat{h}_1(\xi) \overline{\widehat{h}_2(\xi)} d\xi,$$

where  $h_1, h_2 \in H^{s,\varphi}(\mathbb{R})$ . It induces the norm

$$\|h\|_{H^{s,\varphi}(\mathbb{R})} := (h, h)_{H^{s,\varphi}(\mathbb{R})}^{1/2}.$$

Notice that  $H^{s,\varphi}(\mathbb{R})$  is the inner product Hörmander space  $\mathcal{B}_{2,\mu}(\mathbb{R})$  corresponding to the function parameter  $\mu(\xi) := \langle \xi \rangle^s \varphi(\langle \xi \rangle)$  of  $\xi \in \mathbb{R}$  (see the references [12, 13, 37] mentioned above). Therefore  $H^{s,\varphi}(\mathbb{R})$  is a separable Hilbert space embedded continuously in  $\mathcal{S}'(\mathbb{R})$ , and the set  $C_0^\infty(\mathbb{R})$  is dense in  $H^{s,\varphi}(\mathbb{R})$ .

If  $\varphi(r) \equiv 1$ , then  $H^{s,\varphi}(\mathbb{R})$  becomes the Sobolev space  $H^s(\mathbb{R})$  of order  $s$ . Analogously to (3.5), we have the continuous and dense embedding

$$(3.9) \quad H^{s_1}(\mathbb{R}) \hookrightarrow H^{s,\varphi}(\mathbb{R}) \hookrightarrow H^{s_0}(\mathbb{R}) \quad \text{whenever } s_0 < s < s_1, \quad \varphi \in \mathcal{M}.$$

The class of Hilbert function spaces

$$(3.10) \quad \{H^{s,\varphi}(\mathbb{R}) : s \in \mathbb{R}, \varphi \in \mathcal{M}\}$$

is called the refined Sobolev scale over  $\mathbb{R}$  (see [24, Sec. 1.3.3] and [27, Sec. 3.2]).

Using this scale, introduce one-dimensional analogs of the spaces considered above. We let

$$H_+^{s,\varphi}(\mathbb{R}) := \{h \in H^{s,\varphi}(\mathbb{R}) : \text{supp } h \subseteq [0, \infty)\}$$

and interpret  $H_+^{s,\varphi}(\mathbb{R})$  as a (closed) subspace of  $H^{s,\varphi}(\mathbb{R})$ . Then define the normed linear space

$$H_+^{s,\varphi}(0, \tau) := \{h \upharpoonright (0, \tau) : h \in H_+^{s,\varphi}(\mathbb{R})\},$$

$$\|v\|_{H_+^{s,\varphi}(0, \tau)} := \inf \{\|h\|_{H^{s,\varphi}(\mathbb{R})} : h \in H_+^{s,\varphi}(\mathbb{R}), h = v \text{ in } (0, \tau)\},$$

with  $v \in H_+^{s,\varphi}(0, \tau)$ . This space is Hilbert as it is the factor space of  $H_+^{s,\varphi}(\mathbb{R})$  by

$$(3.11) \quad \{h \in H^{s,\varphi}(\mathbb{R}) : \text{supp } h \subseteq \{0\} \cup [\tau, \infty)\}.$$

Both Hilbert spaces  $H_+^{s,\varphi}(\mathbb{R})$  and  $H_+^{s,\varphi}(0, \tau)$  are separable. The set  $C_0^\infty(0, \infty)$  is dense in  $H_+^{s,\varphi}(\mathbb{R})$  [37, Lemma 3.3] so that  $C_+^\infty[0, \tau]$  is dense in  $H_+^{s,\varphi}(0, \tau)$ .

In the Sobolev case of  $\varphi \equiv 1$  we will omit the index  $\varphi$  in the designations of the spaces introduced.

We finish this section with the following observation.

*Remark 3.3.* According to the Sobolev embedding theorem and the above-mentioned result by Agranovich and Vishik [4, theorem 11.1], we obtain the equalities

$$\begin{aligned} C_+^\infty(\overline{\Omega}) &= \bigcap_{\substack{\sigma > \sigma_0, \\ \sigma/(2b) \in \mathbb{Z}}} H_+^{\sigma, \sigma/(2b)}(\Omega), \\ C_+^\infty(\overline{\Omega}) \times (C_+^\infty[0, \tau])^{2m} &= \bigcap_{\substack{\sigma > \sigma_0, \\ \sigma/(2b) \in \mathbb{Z}}} (A, B)(H_+^{\sigma, \sigma/(2b)}(\Omega)). \end{aligned}$$

It follows from them that the mapping (2.5) sets a one-to-one correspondence between the spaces  $C_+^\infty(\overline{\Omega})$  and  $C_+^\infty(\overline{\Omega}) \times (C_+^\infty[0, \tau])^{2m}$ .

#### 4. ABSTRACT AUXILIARY RESULTS

Here we recall the definition of the interpolation with a function parameter in the case of general Hilbert spaces and then discuss the interpolation properties which will be used in Section 5. We follow the monograph [24, Sec. 1.1] (also see [22, Sec. 2]). It is sufficient to restrict ourselves to separable complex Hilbert spaces.

Let  $X := [X_0, X_1]$  be an ordered couple of separable complex Hilbert spaces such that the continuous and dense embedding  $X_1 \hookrightarrow X_0$  holds. This couple is said to be admissible. For  $X$  there exists an isometric isomorphism  $J : X_1 \leftrightarrow X_0$  such that  $J$  is a self-adjoint and positive operator on  $X_0$  with the domain  $X_1$ . The operator  $J$  is uniquely determined by the couple  $X$  and is called the generating operator for  $X$ .

Let  $\psi \in \mathcal{B}$ , where  $\mathcal{B}$  denotes the set of all Borel measurable functions  $\psi : (0, \infty) \rightarrow (0, \infty)$  such that  $\psi$  is bounded on each compact interval  $[a, b]$ , with  $0 < a < b < \infty$ , and that  $1/\psi$  is bounded on every semiaxis  $[a, \infty)$ , with  $a > 0$ .

Consider the operator  $\psi(J)$ , which is defined (and positive) in  $X_0$  as the Borel function  $\psi$  of  $J$ . Denote by  $[X_0, X_1]_\psi$  or simply by  $X_\psi$  the domain of the operator  $\psi(J)$  endowed with the inner product

$$(u_1, u_2)_{X_\psi} := (\psi(J)u_1, \psi(J)u_2)_{X_0}.$$

It induces the norm  $\|u\|_{X_\psi} := \|\psi(J)u\|_{X_0}$ . The space  $X_\psi$  is Hilbert and separable.

A function  $\psi \in \mathcal{B}$  is called an interpolation parameter if the following condition is fulfilled for all admissible couples  $X = [X_0, X_1]$  and  $Y = [Y_0, Y_1]$  of Hilbert spaces and for an arbitrary linear mapping  $T$  given on  $X_0$ : if the restriction of  $T$  to  $X_j$  is

a bounded operator  $T : X_j \rightarrow Y_j$  for each  $j \in \{0, 1\}$ , then the restriction of  $T$  to  $X_\psi$  is also a bounded operator  $T : X_\psi \rightarrow Y_\psi$ .

If  $\psi$  is an interpolation parameter, then we say that the Hilbert space  $X_\psi$  is obtained by the interpolation with the function parameter  $\psi$  of the couple  $X = [X_0, X_1]$  (or, in other words, between the spaces  $X_0$  and  $X_1$ ). In this case the dense and continuous embeddings  $X_1 \hookrightarrow X_\psi \hookrightarrow X_0$  are valid.

It is known that a function  $\psi \in \mathcal{B}$  is an interpolation parameter if and only if  $\psi$  is pseudoconcave in a neighbourhood of  $\infty$ , i.e. there is a concave positive function  $\psi_1(r)$  of  $r \gg 1$  such that both the functions  $\psi/\psi_1$  and  $\psi_1/\psi$  are bounded on some neighbourhood of  $\infty$ . This criterion follows from J. Peetre's description of all interpolation functions for the weighted  $L_p(\mathbb{R}^n)$ -type spaces (see [5, Theorem 5.4.4]). The corresponding proof is given in [24, Sec. 1.1.9].

For us, it is important the next consequence of this criterion [24, Theorem 1.11].

**Proposition 4.1.** *Suppose that a function  $\psi \in \mathcal{B}$  varies regularly of index  $\theta$  at infinity, with  $0 < \theta < 1$ , i.e.*

$$\lim_{r \rightarrow \infty} \frac{\psi(\lambda r)}{\psi(r)} = \lambda^\theta \quad \text{for each } \lambda > 0.$$

*Then  $\psi$  is an interpolation parameter.*

*Remark 4.1.* In the case of power functions this proposition leads us to the classical result by J.-L. Lions and S. G. Krein, which consists in that the function  $\psi(r) \equiv r^\theta$  is an interpolation parameter whenever  $0 < \theta < 1$ . Here the exponent  $\theta$  is regarded as a number parameter of the interpolation.

At the end of this section we formulate two properties of the interpolation; they will be used in our proofs. The first of them enables us to reduce the interpolation of subspaces or factor spaces to the interpolation of initial spaces (see [24, Sec. 1.1.6] and [35, Sec. 1.17]). Note that subspaces are assumed to be closed and that we generally consider nonorthogonal projectors onto subspaces.

**Proposition 4.2.** *Let  $X = [X_0, X_1]$  be an admissible couple of Hilbert spaces, and let  $Y_0$  be a subspace of  $X_0$ . Then  $Y_1 := X_1 \cap Y_0$  is a subspace of  $X_1$ . Suppose that there exists a linear mapping  $P : X_0 \rightarrow X_0$  such that  $P$  is a projector of the space  $X_j$  onto its subspace  $Y_j$  for every  $j \in \{0, 1\}$ . Then the couples  $[Y_0, Y_1]$  and  $[X_0/Y_0, X_1/Y_1]$  are admissible, and*

$$\begin{aligned} [Y_0, Y_1]_\psi &= X_\psi \cap Y_0, \\ [X_0/Y_0, X_1/Y_1]_\psi &= X_\psi / (X_\psi \cap Y_0) \end{aligned}$$

*with equivalence of norms. Here  $\psi \in \mathcal{B}$  is an arbitrary interpolation parameter.*

The second property reduces the interpolation of direct sums of Hilbert spaces to the interpolation of their summands.

**Proposition 4.3.** *Let  $[X_0^{(j)}, X_1^{(j)}]$ , with  $j = 1, \dots, p$ , be a finite collection of admissible couples of Hilbert spaces. Then*

$$\left[ \bigoplus_{j=1}^p X_0^{(j)}, \bigoplus_{j=1}^p X_1^{(j)} \right]_\psi = \bigoplus_{j=1}^p [X_0^{(j)}, X_1^{(j)}]_\psi$$

with equality of norms. Here  $\psi \in \mathcal{B}$  is an arbitrary interpolation parameter.

## 5. PROOF OF THE MAIN RESULT

We will previously prove that the spaces appearing in (2.6) can be obtained by the interpolation with a function parameter between certain Sobolev spaces. Using this interpolation we will deduce Main Theorem from the above-mentioned result by Agranovich and Vishik.

In this section we suppose that

$$(5.1) \quad s, s_0, s_1 \in \mathbb{R}, \quad s_0 < s < s_1, \quad \text{and} \quad \varphi \in \mathcal{M}.$$

Consider the function

$$(5.2) \quad \psi(r) := \begin{cases} r^{(s-s_0)/(s_1-s_0)} \varphi(r^{1/(s_1-s_0)}) & \text{for } r \geq 1, \\ \varphi(1) & \text{for } 0 < r < 1. \end{cases}$$

This function is an interpolation parameter by Proposition 4.1 because  $\psi$  varies regularly of index  $\theta := (s - s_0)/(s_1 - s_0)$  at infinity, with  $0 < \theta < 1$ . We will interpolate couples of Sobolev spaces with the function parameter  $\psi$ .

We begin with anisotropic spaces and prove necessary interpolation formulas for the spaces  $H^{s,s\gamma,\varphi}(\mathbb{R}^2)$ ,  $H_+^{s,s\gamma,\varphi}(\mathbb{R}^2)$ , and  $H_+^{s,s\gamma,\varphi}(\Omega)$  deducing each next formula from the previous one. The corresponding results will be formulated as lemmas.

**Lemma 5.1.** *On the assumption (5.1) we have*

$$(5.3) \quad H^{s,s\gamma,\varphi}(\mathbb{R}^2) = [H^{s_0,s_0\gamma}(\mathbb{R}^2), H^{s_1,s_1\gamma}(\mathbb{R}^2)]_\psi$$

with equality of norms.

*Proof.* The couple of Sobolev spaces

$$X := [H^{s_0,s_0\gamma}(\mathbb{R}^2), H^{s_1,s_1\gamma}(\mathbb{R}^2)]$$

is admissible in view of (3.5). The generating operator for this couple is given by the formula

$$J : w \mapsto \mathcal{F}^{-1}[r_\gamma^{s_1-s_0} \mathcal{F}w], \quad \text{with} \quad w \in H^{s_1,s_1\gamma}(\mathbb{R}^2).$$

This follows immediately from the definition of these spaces. Here  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  stand for the operators of the direct and inverse Fourier transform (in two variables) of tempered distributions given in  $\mathbb{R}^2$ .

Note that  $J$  is reduced to the operator of multiplication by  $r_\gamma^{s_1-s_0}$  with the help of the Fourier transform considered as an isometric isomorphism

$$\mathcal{F} : H^{s_0,s_0\gamma}(\mathbb{R}^2) \leftrightarrow L_2(\mathbb{R}^2, r_\gamma^{2s_0}(\xi, \eta) d\xi d\eta).$$

Hence  $\mathcal{F}$  reduces  $\psi(J)$  to the operator of multiplication by the function

$$\psi(r_\gamma^{s_1-s_0}(\xi, \eta)) \equiv r_\gamma^{s-s_0}(\xi, \eta) \varphi(r_\gamma(\xi, \eta)),$$

in view of (5.2). Now for each  $w \in C_0^\infty(\mathbb{R}^2)$  we may write the following:

$$\begin{aligned} \|w\|_{X_\psi}^2 &= \|\psi(J)w\|_{H^{s_0, s_0\gamma}(\mathbb{R}^2)}^2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi(r_\gamma^{s_1-s_0}(\xi, \eta)) (\mathcal{F}w)(\xi, \eta)|^2 r_\gamma^{2s_0}(\xi, \eta) d\xi d\eta \\ &= \|w\|_{H^{s, s\gamma, \varphi}(\mathbb{R}^2)}. \end{aligned}$$

This implies the equality of spaces (5.3) as  $C_0^\infty(\mathbb{R}^2)$  is dense in both of them. (Note that  $C_0^\infty(\mathbb{R}^2)$  is dense in the second space denoted by  $X_\psi$  because  $C_0^\infty(\mathbb{R}^n)$  is dense in the space  $H^{s_1, s_1\gamma}(\mathbb{R}^2)$  embedded continuously and densely in  $X_\psi$ .)  $\square$

To apply this lemma to the interpolation between the subspaces  $H_+^{s_0, s_0\gamma}(\mathbb{R}^2)$  and  $H_+^{s_1, s_1\gamma}(\mathbb{R}^2)$  we need the following preparatory result.

Let  $\Pi$  be an open half-plain in  $\mathbb{R}^2$  such that its boundary  $\partial\Pi$  is parallel to a certain coordinate axis. The anisotropic Sobolev space  $H^{s, s\gamma}(\Pi)$  is defined as follows

$$\begin{aligned} H^{s, s\gamma}(\Pi) &:= \{w \upharpoonright \Pi : w \in H^{s, s\gamma}(\mathbb{R}^2)\}, \\ \|v\|_{H^{s, s\gamma}(\Pi)} &:= \inf \{\|w\|_{H^{s, s\gamma}(\mathbb{R}^2)} : w \in H^{s, s\gamma}(\mathbb{R}^2), w = v \text{ in } \Pi\}. \end{aligned}$$

This space is Hilbert.

**Lemma 5.2.** *Let numbers  $k \in \mathbb{N}$  and  $\varepsilon > 0$  be arbitrarily chosen. There exists a bounded linear operator  $T_\Pi^{k, \varepsilon} : L_2(\Pi) \rightarrow L_2(\mathbb{R}^2)$  that satisfies the following conditions:*

- (i) *The mapping  $T_\Pi^{k, \varepsilon}$  is an extension operator; i.e.,  $T_\Pi^{k, \varepsilon}v = v$  in  $\Pi$  for each  $v \in L_2(\Pi)$ .*
- (ii) *If  $s, s\gamma \in \mathbb{N} \cap [1, k]$ , then the restriction of  $T_\Pi^{k, \varepsilon}$  to  $H^{s, s\gamma}(\Pi)$  defines a bounded operator*

$$T_\Pi^{k, \varepsilon} : H^{s, s\gamma}(\Pi) \rightarrow H^{s, s\gamma}(\mathbb{R}^2).$$

- (iii) *Let  $E$  be an open interval (bounded or not) that lies on  $\partial\Pi$ , and let  $\nu$  be the unit vector of an inner normal to  $\partial\Pi$  (with respect to  $\Pi$ ). If a function  $v \in L_2(\Pi)$  is equal to zero on the set  $\{x_1 + x_2\nu : x_1 \in E, 0 < x_2 < \varepsilon\}$ , then  $T_\Pi^{k, \varepsilon}v \equiv 0$  on the set  $\{x_1 + x_2\nu : x_1 \in E, x_2 < 0\}$ .*

*Proof.* Without loss of generality we may restrict ourselves to the case when  $\Pi = \{(x, t) : x \in \mathbb{R}, t > 0\}$ . (The general situation is reduced to this case by translation and reflection in the plain.) We construct the operator  $T_\Pi^{k, \varepsilon}$  with the help of the extension method by M. R. Hestenes (see [6, Sec. 9.9] or [35, Sec. 2.9.1]).

Namely, given a function  $v : \bar{\Pi} \rightarrow \mathbb{C}$ , let

$$(T_{\Pi}^{k,\varepsilon} v)(x, t) := \begin{cases} v(x, t) & \text{for } x \in \mathbb{R}, t \geq 0, \\ \chi_{\varepsilon}(t) \sum_{j=1}^{k+1} \lambda_j v(x, -t/j) & \text{for } x \in \mathbb{R}, t < 0. \end{cases}$$

Here the numbers  $\lambda_1, \dots, \lambda_k, \lambda_{k+1}$  are chosen so that

$$\sum_{j=1}^{k+1} \lambda_j \left( -\frac{1}{j} \right)^{\alpha} = 1, \quad \alpha = 0, 1, \dots, k.$$

Moreover,  $\chi_{\varepsilon} \in C^{\infty}(\mathbb{R})$  is a fixed function such that  $\chi_{\varepsilon}(t) = 1$  if  $t > -\varepsilon/3$  and that  $\chi_{\varepsilon}(t) = 0$  if  $t < -2\varepsilon/3$ . Then  $v \in C^k(\bar{\Pi})$  implies  $T_{\Pi}^{(k)} v \in C^k(\mathbb{R}^2)$ .

Evidently, the mapping  $v \mapsto T_{\Pi}^{k,\varepsilon} v$  defines a bounded linear operator  $T_{\Pi}^{k,\varepsilon} : L_2(\Pi) \rightarrow L_2(\mathbb{R}^2)$  that complies with conditions (i) and (iii). According to [6, Sec. 9.9] this operator satisfies condition (ii) as well.  $\square$

**Lemma 5.3.** *In addition to (5.1) suppose that all the numbers  $s_0, s_1, s_0\gamma$ , and  $s_1\gamma$  are positive integers. Then*

$$(5.4) \quad H_+^{s,s\gamma,\varphi}(\mathbb{R}^2) = [H_+^{s_0,s_0\gamma}(\mathbb{R}^2), H_+^{s_1,s_1\gamma}(\mathbb{R}^2)]_{\psi},$$

$$(5.5) \quad H_+^{s,s\gamma,\varphi}(\Omega) = [H_+^{s_0,s_0\gamma}(\Omega), H_+^{s_1,s_1\gamma}(\Omega)]_{\psi}$$

with equivalence of norms.

*Proof.* First deduce (5.4). Let  $\Pi := \{(x, t) : x \in \mathbb{R}, t < 0\}$ , and let  $T_{\Pi}^{s_1,1}$  be the extension operator from Lemma 5.2. The mapping  $P : w \mapsto w - T_{\Pi}^{s_1,1}(w \upharpoonright \Pi)$ , where  $w \in L_2(\mathbb{R}^2)$ , defines the projector of the space  $H_+^{s_j,s_j\gamma}(\mathbb{R}^2)$  onto its subspace  $H_+^{s_j,s_j\gamma}(\mathbb{R}^2)$  for every  $j \in \{0, 1\}$ . Therefore by Proposition 4.2 and Lemma 5.1 we may write

$$\begin{aligned} [H_+^{s_0,s_0\gamma}(\mathbb{R}^2), H_+^{s_1,s_1\gamma}(\mathbb{R}^2)]_{\psi} &= [H^{s_0,s_0\gamma}(\mathbb{R}^2), H^{s_1,s_1\gamma}(\mathbb{R}^2)]_{\psi} \cap H_+^{s_0,s_0\gamma}(\mathbb{R}^2) \\ &= H^{s,s\gamma,\varphi}(\mathbb{R}^2) \cap H_+^{s_0,s_0\gamma}(\mathbb{R}^2) = H_+^{s,s\gamma,\varphi}(\mathbb{R}^2) \end{aligned}$$

up to equivalence of norms. Formula (5.4) is proved.

Now we will deduce (5.5) from (5.4). To this end we construct a certain projector  $P_0$  of each space  $H_+^{s_j,s_j\gamma}(\mathbb{R}^2)$ , with  $j \in \{0, 1\}$ , onto its subspace  $H_Q^{s_j,s_j\gamma}(\mathbb{R}^2)$  defined by (3.8). Consider the half-plains

$$\Pi_1 := \{(x, t) : x \in \mathbb{R}, t < \tau\},$$

$$\Pi_2 := \{(x, t) : x < l, t \in \mathbb{R}\},$$

$$\Pi_3 := \{(x, t) : x > 0, t \in \mathbb{R}\}.$$

For every  $\alpha \in \{1, 2, 3\}$ , let  $R_{\alpha}$  denote the restriction mapping  $w \mapsto w \upharpoonright \Pi_{\alpha}$ , with  $w \in L_2(\mathbb{R}^2)$ , and let  $T_{\alpha}$  denote the extension operator  $T_{\Pi_{\alpha}}^{s_1,l}$  from Lemma 5.2. Consider the mapping  $P_0 : w \mapsto w - \Lambda w$ , with  $w \in H_+^{s_0,s_0\gamma}(\mathbb{R}^2)$  and  $\Lambda w := T_3 R_3 T_2 R_2 T_1 R_1 w$ . It follows from lemma 5.2 that  $P_0$  is the projector required. Indeed,  $P_0$  is a linear

bounded operator on  $H_+^{s_j, s_j\gamma}(\mathbb{R}^2)$  for every  $j \in \{0, 1\}$ . Moreover, if  $w = 0$  in  $\Omega$ , then  $\Lambda w = 0$  in  $\mathbb{R}^2$ ; therefore  $P_0 w = w$  for each  $w \in H_Q^{s_j, s_j\gamma}(\mathbb{R}^2)$ .

Since the projector  $P_0$  is given, we may apply Proposition 4.2 and formula (5.4) and write

$$\begin{aligned} & [H_+^{s_0, s_0\gamma}(\Omega), H_+^{s_1, s_1\gamma}(\Omega)]_\psi \\ &= [H_+^{s_0, s_0\gamma}(\mathbb{R}^2)/H_Q^{s_0, s_0\gamma}(\mathbb{R}^2), H_+^{s_1, s_1\gamma}(\mathbb{R}^2)/H_Q^{s_1, s_1\gamma}(\mathbb{R}^2)]_\psi \\ &= [H_+^{s_0, s_0\gamma}(\mathbb{R}^2), H_+^{s_1, s_1\gamma}(\mathbb{R}^2)]_\psi / ([H_+^{s_0, s_0\gamma}(\mathbb{R}^2), H_+^{s_1, s_1\gamma}(\mathbb{R}^2)]_\psi \cap H_Q^{s_0, s_0\gamma}(\mathbb{R}^2)) \\ &= H_+^{s, s\gamma, \varphi}(\mathbb{R}^2) / (H_+^{s, s\gamma, \varphi}(\mathbb{R}^2) \cap H_Q^{s_0, s_0\gamma}(\mathbb{R}^2)) = H_+^{s, s\gamma, \varphi}(\mathbb{R}^2)/H_Q^{s, s\gamma, \varphi}(\mathbb{R}^2) \\ &= H_+^{s, s\gamma, \varphi}(\Omega) \end{aligned}$$

up to equivalence of norms. Formula (5.5) is proved.  $\square$

It remains to prove a necessary interpolation formula for the space  $H_+^{s, \varphi}(0, \tau)$ .

**Lemma 5.4.** *In addition to (5.1) suppose that  $s_0 \geq 0$ . Then*

$$(5.6) \quad H_+^{s, \varphi}(0, \tau) = [H_+^{s_0}(0, \tau), H_+^{s_1}(0, \tau)]_\psi$$

with equivalence of norms.

*Proof.* The formula (5.6) can be deduced by analogy with the anisotropic spaces case considered in the previous lemmas. For the sake of the argumentation completeness, let us give the proof.

First note that an analog of Lemma 5.1 for isotropic spaces over  $\mathbb{R}^n$  is proved in [22, Sec. 3.2, Theorem 3.4] (also see [24, Sec. 1.3.4, Theorem 1.14]). Specifically,

$$(5.7) \quad H^{s, \varphi}(\mathbb{R}) = [H^{s_0}(\mathbb{R}), H^{s_1}(\mathbb{R})]_\psi$$

with equality of norms.

To deduce (5.6) from (5.7) we will apply the following one-dimensional analog of Lemma 5.2 on extension operator. Let  $G \subset \mathbb{R}$  be an open semiaxis and  $k \in \mathbb{N}$ . Then there exists a bounded linear operator  $T_G^{(k)} : L_2(G) \rightarrow L_2(\mathbb{R})$  such that  $T_G^{(k)} v$  is an extension of  $v \in L_2(G)$  and that the mapping  $v \mapsto T_G^{(k)} v$  defines the bounded operator  $T_G^{(k)} : H^s(G) \rightarrow H^s(\mathbb{R})$  for every real  $s \in [0, k)$ . Here, as usual,

$$\begin{aligned} H^s(G) &:= \{h \upharpoonright G : h \in H^s(\mathbb{R})\}, \quad \text{with} \\ \|v\|_{H^s(G)} &:= \inf \{ \|h\|_{H^s(\mathbb{R})} : h \in H^s(\mathbb{R}), h = v \text{ in } G \}, \end{aligned}$$

is the Sobolev space over  $G$  of order  $s$ . This analog is a special case of Lemma 2.9.3 from [35]. As above, the operator  $T_G^{(k)}$  can be constructed with the help of the extension method by M. R. Hestenes.

Chose  $k \in \mathbb{N}$  so that  $s_1 < k$ . The mapping  $P : h \mapsto h - T_G^{(k)}(h \upharpoonright G)$ , where  $h \in L_2(\mathbb{R})$  and  $G := (-\infty, 0)$ , defines the projector of the space  $H^{s_j}(\mathbb{R})$  onto its

subspace  $H_+^{s_j}(\mathbb{R})$  for every  $j \in \{0, 1\}$ . Therefore by Proposition 4.2 and formula (5.7) we may write

$$(5.8) \quad [H_+^{s_0}(\mathbb{R}), H_+^{s_1}(\mathbb{R})]_\psi = [H^{s_0}(\mathbb{R}), H^{s_1}(\mathbb{R})]_\psi \cap H_+^{s_0}(\mathbb{R}) = H_+^{s, \varphi}(\mathbb{R})$$

up to equivalence of norms.

Now let us deduce (5.6) from (5.8). Recall that  $H_+^{s, \varphi}(0, \tau)$  is the factor space of the space  $H_+^{s, \varphi}(\mathbb{R})$  by its subspace (3.11). The latter coincides with

$$H_{[\tau, \infty)}^{s, \varphi}(\mathbb{R}) := \{h \in H^{s, \varphi}(\mathbb{R}) : \text{supp } h \subseteq [\tau, \infty)\}$$

because  $s > 0$ . The mapping  $P_\tau : h \mapsto h - T_{G_\tau}^{(k)}(h \upharpoonright G_\tau)$ , where  $h \in L_2(\mathbb{R})$  and  $G_\tau := (-\infty, \tau)$ , sets the projector of the space  $H_+^{s_j}(\mathbb{R})$  onto its subspace  $H_{[\tau, \infty)}^{s_j}(\mathbb{R})$  for every  $j \in \{0, 1\}$ . Therefore by Proposition 4.2 and formula (5.8) we may write

$$\begin{aligned} [H_+^{s_0}(0, \tau), H_+^{s_1}(0, \tau)]_\psi &= [H_+^{s_0}(\mathbb{R})/H_{[\tau, \infty)}^{s_0}(\mathbb{R}), H_+^{s_1}(\mathbb{R})/H_{[\tau, \infty)}^{s_1}(\mathbb{R})]_\psi \\ &= [H_+^{s_0}(\mathbb{R}), H_+^{s_1}(\mathbb{R})]_\psi / ([H_+^{s_0}(\mathbb{R}), H_+^{s_1}(\mathbb{R})]_\psi \cap H_{[\tau, \infty)}^{s_0}(\mathbb{R})) \\ &= H_+^{s, \varphi}(\mathbb{R})/H_{[\tau, \infty)}^{s, \varphi}(\mathbb{R}) = H_+^{s, \varphi}(0, \tau) \end{aligned}$$

up to equivalence of norms. Formula (5.6) is proved.  $\square$

Now we may give

*The proof of Main Theorem.* Let  $\sigma > \sigma_0$  and  $\varphi \in \mathcal{M}$ . Choose a number  $\sigma_1 \in \mathbb{N}$  so that  $\sigma_1/(2b) \in \mathbb{N}$  and  $\sigma_1 > \sigma$ . According to M. S. Agranovich and M. I. Vishik [4, Theorem 11.1], the mapping (2.5) extends uniquely to isomorphisms between Sobolev spaces

$$(5.9) \quad (A, B) : H_+^{\sigma_k, \sigma_k/(2b)}(\Omega) \leftrightarrow \mathcal{H}_k \quad \text{for every } k \in \{0, 1\},$$

where

$$\mathcal{H}_k := H_+^{\sigma_k - 2m, (\sigma_k - 2m)/(2b)}(\Omega) \oplus \bigoplus_{j=1}^m (H_+^{(\sigma_k - m_j - 1/2)/(2b)}(0, \tau))^2.$$

Define an interpolation parameter by the formula

$$\psi(r) := \begin{cases} r^{(\sigma - \sigma_0)/(\sigma_1 - \sigma_0)} \varphi(r^{1/(\sigma_1 - \sigma_0)}) & \text{for } r \geq 1, \\ \varphi(1) & \text{for } 0 < r < 1, \end{cases}$$

which is analogous to (5.2). Applying the interpolation with the function parameter  $\psi$  to (5.9), we get another isomorphism

$$(5.10) \quad (A, B) : [H_+^{\sigma_0, \sigma_0/(2b)}(\Omega), H_+^{\sigma_1, \sigma_1/(2b)}(\Omega)]_\psi \leftrightarrow [\mathcal{H}_0, \mathcal{H}_1]_\psi.$$

This isomorphism is a unique extension by continuity of the mapping (2.5) because  $C_+^\infty(\overline{\Omega})$  is dense in the domain of (5.10).

Let us describe the interpolation spaces appearing in (5.10). According to Lemma 5.3 we have

$$[H_+^{\sigma_0, \sigma_0/(2b)}(\Omega), H_+^{\sigma_1, \sigma_1/(2b)}(\Omega)]_\psi = H_+^{\sigma, \sigma/(2b), \varphi}(\Omega)$$

with equality of norms. Next, applying Proposition 4.3 and Lemmas 5.3 and 5.4 we may write

$$\begin{aligned} [\mathcal{H}_0, \mathcal{H}_1]_\psi &= [H_+^{\sigma_0-2m, (\sigma_0-2m)/(2b)}(\Omega), H_+^{\sigma_1-2m, (\sigma_1-2m)/(2b)}(\Omega)]_\psi \\ &\oplus \bigoplus_{j=1}^m ([H_+^{(\sigma_0-m_j-1/2)/(2b)}(0, \tau), H_+^{(\sigma_1-m_j-1/2)/(2b)}(0, \tau)]_\psi)^2 \\ &= H_+^{\sigma-2m, (\sigma-2m)/(2b), \varphi}(\Omega) \oplus \bigoplus_{j=1}^m (H_+^{(\sigma-m_j-1/2)/(2b), \varphi}(0, \tau))^2 \end{aligned}$$

with equality of norms. Note that the function  $\psi$  satisfies (5.2) because the parameters  $s_0$ ,  $s_1$ , and  $s$  in these lemmas differ from  $\sigma_0$ ,  $\sigma_1$ , and  $\sigma$  respectively in the same magnitude. Thus, the isomorphism (5.10) becomes (2.6).  $\square$

## 6. FINAL REMARKS

Main Theorem can be used to investigate regularity of solutions to parabolic problems. Specifically, applying Hörmander's Embedding Theorem [12, Theorem 2.2.7], we may establish sufficient conditions for the weak solution to be classical (compare with [20, Sec. 5 and 6] or [24, Sec. 4.1.2], where elliptic boundary-value problems are considered).

The investigation of parabolic initial-boundary value problems with nonhomogeneous initial conditions can be reduced to the case of homogeneous ones (see [4, § 10] in the case of Sobolev spaces). In this connection, we also mention J.-L. Lions and E. Magenes' approach [17, Sec. 6.4] based on interpolation with a number parameter. Apparently, their methods may admit a generalization to the case of function interpolation parameters.

An analog of Main Theorem is also true for the many-dimensional case, when the parabolic problem is given in a cylinder situated in  $\mathbb{R}^{n+1}$ , with  $n \geq 2$ . This analog can be deduced from M. S. Agranovich and M. I. Vishik's result [4, Theorem 11.1] by means of interpolation with a function parameter.

The above-mentioned applications and generalizations of Main Theorem will be published elsewhere.

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